Exercice n°1 Inverser

$$A = \begin{pmatrix} 5 & 3 & -2 \\ -6 & -4 & 3 \\ -3 & -1 & 1 \end{pmatrix} \Bigg\| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_1 \leftarrow L_1 + L_2 \begin{pmatrix} -1 & -1 & 1 \\ -6 & -4 & 3 \\ -3 & -1 & 1 \end{pmatrix} \Bigg\| \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_2 \leftarrow L_2 - 2L_3 \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ 3 & -1 & 1 \end{pmatrix} \Bigg\| \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_1 \leftarrow L_1 - L_2 \begin{pmatrix} -1 & +1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \Bigg\| \begin{pmatrix} & & \\ & & \end{pmatrix}$$

2^e exercice :

$L_1 \leftarrow L_1 + L_2$

$$A = \begin{pmatrix} 5 & 3 & -2 \\ -6 & -4 & 3 \\ -3 & -1 & 1 \end{pmatrix} \Leftrightarrow \begin{cases} 5x + 3y - 2z = X \\ -6x - 4y + 3z = Y \\ -3x - y + z = Z \end{cases} \Leftrightarrow \begin{cases} -x - y + z = X - Y \\ -6x - 4y + 3z = Y \\ -3x - y + z = Z \end{cases}$$

$$\Leftrightarrow \begin{cases} z = X - Y + x + y \\ -6x - 4y + 3z = Y \\ -3x - y + z = Z \end{cases} \Leftrightarrow \begin{cases} -x - y + X - Y + x + y = X - Y \\ -6x - 4y + 3(X - Y + x + y) = Y \\ -3x - \cancel{y} + X - Y + x + \cancel{y} = Z \end{cases}$$

$$\Leftrightarrow \begin{cases} z = X - Y + x + y \\ -6x - 4y + 3X - 3Y + 3x + 3y = Y \\ -2x = Z + Y - X \end{cases}$$

$$\Leftrightarrow \begin{cases} z = X - Y + \dfrac{-Z - Y + X}{2} + y \\ -3\left(\dfrac{-Z - Y + X}{2}\right) - y = 3X + 4Y \\ x = \dfrac{-Z - Y + X}{2} \end{cases} \Leftrightarrow \begin{cases} z = \dfrac{3X - 3Y - Z}{2} \\ y = \dfrac{3Z - 5Y - 9X}{2} \\ x = \dfrac{-Z - Y + X}{2} \end{cases}$$

$$\begin{cases} x = \frac{-1}{2}Z - \frac{1}{2}Y + \frac{1}{2}X \\ y = \frac{3}{2}Z - \frac{5}{2}Y - \frac{9}{2}X \\ z = \frac{3}{2}X - \frac{3}{2}Y - \frac{1}{2}Z \end{cases}$$

✓

Correction : $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ 3 & 2 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 3 & 1 & 3 \\ 6 & 4 & 2 \end{pmatrix}$.

Methode : Verif Matrice Inv : $A \times A^{-1} = Id$.

Exercice 2 : $f : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$.

$P(x) \longmapsto XP''(x) + 2P(x)$

$g : \mathbb{R}^3 \longmapsto M_2(\mathbb{R})$

$(x, y, 3) \rightarrow \begin{pmatrix} x+y & y+3 \\ x+3 & 0 \end{pmatrix}$.

Matrices dans Ⓑ

Maths

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$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & 2 \\ 1 & -2 & -1 \end{pmatrix} \qquad Au = v \Longleftrightarrow u = A^{-1} v$$

Soient $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ et $v = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

$$Au = v \Longleftrightarrow \begin{cases} x - y - z = X \\ -x + 2z = Y \\ x - 2y - z = Z \end{cases} \Longleftrightarrow \begin{matrix} \\ L_2 + 3L_1 \\ L_3 - 2L_1 \end{matrix} \begin{cases} x - y - z = X \\ 2x - z = Y + 3X \\ -x + 2 = -2 - 2X \end{cases}$$

$$\Longleftrightarrow \begin{matrix} \\ L_3 + L_2 \\ \\ \end{matrix} \begin{cases} x - y - z = X \\ 2x - 2 = Y + 3X \\ x = X + Y + 2 \end{cases} \Longleftrightarrow \begin{cases} y = X - 2 \\ z = 2x - y - 3X = X + Y + 2Z \\ x = X + Y + Z. \end{cases}$$

$$\Longleftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X + Y + Z \\ X - Z \\ X + Y + 2Z \end{pmatrix} \Longleftrightarrow U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 1 \\ -1 & 3 & -3 \end{pmatrix} \qquad Bu = v \Longleftrightarrow \begin{cases} x - 2y + 2z = X \\ -x + y + z = Y \\ -x + 3y - 3z = Z \end{cases}$$

$$\Longleftrightarrow \begin{cases} -y + 3z = X + Y \\ y - z = Z + X \\ x - 2y + 2z = X \end{cases} \Longleftrightarrow \begin{cases} x - 2y + 2z = X \\ -y + 3z = X + X \\ 2z = 2X + Y + Z \end{cases}$$

$$\Longleftrightarrow \begin{cases} y = 3z - X - Y = 2X + \frac{1}{2}Y + \frac{3}{2}Z \\ z = X + \frac{1}{2}Y + \frac{1}{2}Z \\ x = X + 2y - 2z = 2X + 2Z \end{cases}$$

$$\Longleftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 0 & 2 \\ 2 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cancel{X} \\ Y \\ Z \end{pmatrix}$$

Maths

Exercice n°6:

$E = \mathbb{R}_2[X]$

$f : E \longrightarrow E$

$P \longmapsto 2(X+1)P - (X^2+1)P'$ (linéaire)

$\mathcal{B} = (1, X, X^2)$

$\mathcal{B}' = (1, X-1, (X+1)^2)$

$$1)\ A = \mathrm{Mat}_{\mathcal{B}}(f) = \begin{array}{cc} \begin{matrix} f(1) & f(X) & f(X^2) \end{matrix} & \\ \begin{pmatrix} 2 & -1 & 0 \\ 2 & 2 & -2 \\ 0 & 1 & 2 \end{pmatrix} & \begin{matrix} 1 \\ X \\ X^2 \end{matrix} \end{array}$$

$f(1) = 2X + 2$

$f(X) = 2(X+1)X - (X^2+1)\times 1$

$= 2(X^2+X) - (X^2+1)$

$= X^2 + 2X \quad -1$

$f(X^2) = 2(X+1)X^2 - (X^2+1)2X$

$= 2X^2 \quad -2X$

$2X^2 + 4X - 2 \quad 2$

$-6X + 6 \quad -6$

4

$$3)\ B = \mathrm{Mat}_{\mathcal{B}'}(f) = \begin{array}{cc} \begin{matrix} f(1) & f(X) & f(X^2) \end{matrix} & \\ \begin{pmatrix} 4 & -2 & -2 \\ 2 & 0 & -6 \\ 0 & 1 & 2 \end{pmatrix} & \begin{matrix} 1 \\ X-1 \\ (X+1)^2 \end{matrix} \end{array}$$

$X^2 + 2X + 1$

$2X^2 + 4X + 2$

$-6X + 6$

2 bis) $B =$ $f(1) = 2X + 2$

$f(X-1) = 2(X+1)(X-1) - (X^2+1)$

$= 2X^2 - 2 - X^2 - 1$

$= X^2 - 3$

$f((X+1)^2 = 2(X+1)^3 - 2(X^2+1)(X+1)$

$= 2X^3 + 6X^2 + 6X + 2 - 2X^3 - 2X^2 - 2X - 2$

$= 4X^2 + 4X.$

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4) $\mathrm{Mat}_{B',B}(f) = \begin{array}{cccc} f(1) & f(X-1) & f((X+1)^2) & \\ \left(\begin{array}{c} 2 \\ 2 \\ 0 \end{array}\right. & \begin{array}{c} -3 \\ 0 \\ 1 \end{array} & \left.\begin{array}{c} 0 \\ 4 \\ 4 \end{array}\right) & \begin{array}{c} 1 \\ X \\ X^2 \end{array} \end{array}$

2) $\underbrace{2X}+2 = \underbrace{2(X-1)+2}+2$

$= 2(X-1)+4$

$\underbrace{X^2}-3 = \underbrace{(X+1)^2-2X-1}-3$

$= (X+1)^2 - 2X - 4 = (X+1)^2 - 2(X-1) - 2 - 4$

$\underbrace{4X^2}+4X = \underbrace{4(X+1)^2-8X-4}+4X$

$= 4(X+1)^2 - 4X - 4$

$= 4(X+1)^2 - 4(X-1) - 4 - 4$

$\mathrm{Mat}_{B'}(f) = \begin{pmatrix} 4 & -6 & -8 \\ 2 & -2 & -4 \\ 0 & 2 & 4 \end{pmatrix}$

5) $P = \mathrm{Mat}_{B,B'}(id) = \begin{array}{cccc} id(1) & id(X-1) & id((X+1)^2). & \\ \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} -1 \\ 1 \\ 0 \end{array} & \left.\begin{array}{c} 1 \\ 2 \\ 1 \end{array}\right) & \begin{array}{c} 1 \\ X \\ X^2 \end{array} \end{array}$

$Q = \mathrm{Mat}_{B',B}(id) = \begin{array}{cccc} id(1) & id(X) & id(X^2) & \\ \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 1 \\ 1 \\ 0 \end{array} & \left.\begin{array}{c} -5 \\ -2 \\ 1 \end{array}\right) & \begin{array}{c} 1 \\ X-1 \\ (X+1)^2 \end{array} \end{array}$

6) $\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

Soient $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $v = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

$Pu = v \Leftrightarrow \begin{cases} x - y + z = X \\ y + 2z = Y \\ z = Z \end{cases} \Leftrightarrow \begin{cases} x = X + Y - 2Z - Z = X + Y - 3Z \\ y = Y - 2Z \\ z = Z \end{cases}$

$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = Q.$

7) $(P^{-1}A)P = P^{-1}(AP).$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 0 \\ 2 & 2 & -2 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Exercice n° 5 : 1) $A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}$ Soient $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ et $v = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$.

$AU = V \Longrightarrow A^{-1}V = U$

$$\Longleftrightarrow \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longrightarrow \begin{cases} x + y - 2z = X \\ x - y + z = Y \\ -2x + y - z = Z \end{cases}$$

$$\Longrightarrow \begin{cases} x + y - 2z = X \\ x - y + z = Y \\ x = -Y - Z \end{cases} \quad 3+2 \qquad \underset{\substack{L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - L_1}}{\Longleftrightarrow} \begin{cases} x + y - 2z = X \\ 2x - z = Y + X \\ -3x + z = Z - X \end{cases}.$$

$$\underset{L_3 \leftarrow L_3 + L_2}{\Longleftrightarrow} \begin{cases} x + y - 2z = X \\ 2x - z = Y + X \\ -x = Y + Z \end{cases} \Longleftrightarrow \begin{cases} y = X - x + 2z = X + Y + Z - 2X \\ -6Y - 4Z \\ z = 2x - Y - X = -2Y - 2Z - Y - X \\ = -X - 3Y - 2Z \\ x = -Y - Z \end{cases}$$

$$\Longleftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -Y - Z \\ -X - 5Y - 3Z \\ -X - 3Y - 2Z \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -5 & -3 \\ -1 & -3 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

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2) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+y-2z \\ x-y+z \\ -2x+y-z \end{pmatrix}$$

Cours : En dimension finie E, B

f bijective $\Leftrightarrow \mathrm{Mat}_B(f)$ inversible

Dans ce cas-là

$$\mathrm{Mat}_B(f^{-1}) = [\mathrm{Mat}_B(f)]^{-1}$$

Ici soit $B = (\vec{i}, \vec{j}, \vec{k})$ base canonique de $\mathbb{R}^3$

$$\mathrm{Mat}_B(f) = \begin{array}{c} \begin{array}{ccc} f(\vec{i}) & f(\vec{j}) & f(\vec{k}) \end{array} \\ \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} \end{array} \begin{array}{c} \vec{i} \\ \vec{j} \\ \vec{k} \end{array} = A$$

Comme A est inversible, f est bijective et

$$\mathrm{Mat}_B(f^{-1}) = \underbrace{\begin{pmatrix} 0 & -1 & -1 \\ -1 & -5 & -3 \\ -1 & -3 & -2 \end{pmatrix}}_{A^{-1}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

CCL : $f^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ -x-5y-3z \\ -x-3y-2z \end{pmatrix}.$$

Exercice n°8 :

$f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$

$P(X) \longmapsto P(X) - X P'(X)$

$B = (1, X, X^2)$

$$\mathrm{Mat}_B(f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$g : \mathbb{R}_3[X] \longrightarrow \mathbb{R}^3$

$P(X) \longrightarrow (P(-1), P(0), P(1))$

$B_1 = (1, X, X^2, X^3)$ $B_2 = (\vec{i}, \vec{j}, \vec{k})$

$$\mathrm{Mat}_B(g) = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercice n° 9

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^{E_{11}} + b\overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}^{E_{12}} + c\overbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}^{E_{21}} + d\overbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}^{E_{22}}$$

$\Rightarrow \mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$ engendre $M_2(\mathbb{R})$. On vérifie facilement qu'elle est libre. CCL : $\mathcal{B}$ base de $M_2(\mathbb{R})$ appelée base canonique.

$$f : \mathcal{M}_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

* Montrons que $f \in \mathcal{L}(\mathcal{M}_2(\mathbb{R}))$

Soient $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$, $\lambda \in \mathbb{R}$.

$$f(\lambda A + A') = f\left(\begin{pmatrix} \lambda a + a' & \lambda b + b' \\ \lambda c + c' & \lambda d + d' \end{pmatrix}\right) = \begin{pmatrix} \lambda d + d' & -\lambda c + c' \\ -\lambda b - b' & \lambda a + a' \end{pmatrix}$$

$$= \lambda\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} + \begin{pmatrix} d' & -c' \\ -b' & a' \end{pmatrix} = \lambda f(A) + f(A')$$

$$\mathcal{B} = \left(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{E_{11}}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E_{12}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_{21}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_{22}}\right).$$

$$\mathrm{Mat}_{\mathcal{B}}(f) = \begin{array}{cccc|c} f(E_{11}) & f(E_{12}) & f(E_{21}) & f(E_{22}) & \\ 0 & 0 & 0 & 1 & E_{11} \\ 0 & 0 & -1 & 0 & E_{12} \\ 0 & -1 & 0 & 0 & E_{21} \\ 1 & 0 & 0 & 0 & E_{22} \end{array}$$

Exercice n°10 :

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Exercice n°10 : 1) $f : \begin{cases} \mathbb{R}_2[X] \to \mathbb{R}_2[X] \\ P(X) \mapsto (X^2-1)P(2) + 2XP(3) \end{cases}$

$\mathcal{B} = (1, X, X^2)$

$$\mathrm{Mat}_{\mathcal{B}}(f) \quad \begin{array}{c} \begin{array}{ccc} f(1) & f(X) & f(X^2) \end{array} \\ \left(\begin{array}{ccc} -1 & -2 & -4 \\ 2 & 6 & 18 \\ 1 & 2 & 4 \end{array}\right) \end{array} \begin{array}{c} \\ 1 \\ X \\ X^2 \end{array}$$

$X^2 - 1 + 2X$

$2X^2 - 2 + 4X$

$4X^2 - 4 + 18X$

2) $g : \begin{cases} \mathbb{R}_3[X] \to \mathbb{R}_3[X] \\ P(X) \mapsto P(X+1) \end{cases}$ $\qquad h : \begin{cases} \mathbb{R}_3[X] \to \mathbb{R}_3[X] \\ P(X) \mapsto P(X-1) \end{cases}$

$\mathcal{B} = (1, X, X^2, X^3)$ $\qquad \mathcal{B} = (1, X, X^2, X^3)$.

$$\mathrm{Mat}_{\mathcal{B}}(g) = \begin{array}{c} \begin{array}{cccc} f(1) & f(X) & f(X^2) & f(X^3) \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ 1 \\ X \\ X^2 \\ X^3 \end{array}$$

$(X+1)^2 = X^2 + 2X + 1$

$(X+1)^3 = (X^2+2X+1)(X+1)$

$= X^3 + 2X^2 + X + X^2 + 2X + 1$

$= X^3 + 3X^2 + 3X + 1$.

$$\mathrm{Mat}_{\mathcal{B}}(h) = \left(\begin{array}{cccc} 1 & -1 & -1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} 1 \\ X \\ X^2 \\ X^3 \end{array}$$

$(X-1)^2 = X^2 - 2X + 1$

$(X-1)^3 = (X^2-2X+1)(X-1)$

$= X^3 - 2X^2 + X - X^2 + 2X$

$+ 1$

b) $g \circ h(P(X)) = g(P(X-1)) = P(X)$.

$h \circ g(P(X)) = h(P(X+1)) = (P(X))$

$\Rightarrow g \circ h = h \circ g = id$

$\Rightarrow h = g^{-1}$

$\Rightarrow \mathrm{Mat}_{\mathcal{B}}(g^{-1}) = \mathrm{Mat}_{\mathcal{B}}(h)$.

3) $u . \mathbb{R}^3 \to \mathbb{R}^3$

$(x, y, z) \mapsto (x + 2y + z, y + 2z)$.

$v . \mathbb{R}^2 \to \mathbb{R}^3$

$(x, y) \mapsto (3x + y, x, 2y)$.

$\mathcal{B} = ((1, 0, 0), (0, 1, 0), (0, 0, 1))$.

$\mathrm{Mat}_{B,B'}(u) = \begin{pmatrix} (1, 0) \\ (2, 2) \\ (0, 2) \end{pmatrix}$ $\qquad$ $\mathrm{Mat}_{B'}(v) = \begin{pmatrix} (3, 1, 0) \\ (1, 0, 2) \\ \ \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$v \circ u \begin{pmatrix} x \\ y \\ z \end{pmatrix} = v \begin{pmatrix} x + 2y + z \\ y + 2z \end{pmatrix} = \begin{pmatrix} 3x + 6y + 3z + y + 2z \\ x + 2y + z \\ 2y + 4z \end{pmatrix} = \begin{pmatrix} 3x + 7y + 5z \\ x + 2y + z \\ 2y + 4z . \end{pmatrix}$

$\Rightarrow \mathrm{Mat}_{B}'(v \circ u) = \begin{array}{c|c|c|c} v\circ u(\vec{i}) & v\circ u(\vec{j}) & v\circ u(\vec{k}) & \\ 3 & 7 & 5 & \vec{i} \\ 1 & 2 & 1 & \vec{j} \\ 0 & 2 & 4 & \vec{k} \end{array} = MN.$

$\mathrm{Mat}_{B}(u \circ v) = NM = \begin{pmatrix} 5 & 3 \\ 1 & 4 \end{pmatrix}$

Exercice n° 11 :

$V(P) = (P(x_0), P(x_1), \ldots, P(x_n))$.

$\mathcal{B} = (1, X, \ldots, X^n)$ bc de $\mathbb{R}_n[X]$.

$\mathcal{B}' = ((\underbrace{1, 0, \ldots, 0}_{n+1}), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1))$ bc de $\mathbb{R}^{n+1}$

$\mathrm{Mat}_{B,B'}(V) = \begin{pmatrix} 1 & x_0 & x_0^2 & \text{—} & x_0^n \\ | & x_1 & | & & | \\ | & | & | & & | \\ 1 & x_n & x_n^2 & \text{—} & x_n^n \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ | \\ e_{n+1} \end{matrix}$

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Exercice n°12: 1)

Exercice Inverser $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Longleftrightarrow \begin{cases} y + z = X \\ x + z = Y \\ x + y = Z \end{cases} \underset{L_3 = L_3 - L_1}{\Longleftrightarrow} \begin{cases} y + z = X \\ x + z = Y \\ x - z = Z - X . \end{cases}$

$\underset{L_3 \leftarrow L_3 + L_2}{\Longleftrightarrow} \begin{cases} y + z = X \\ x + z = Y \\ 2x = Z - X + Y \end{cases} \Longleftrightarrow \begin{cases} y = \frac{1}{2}X - \frac{1}{2}Y + \frac{1}{2}Z \\ z = Y - z = \frac{1}{2}X + \frac{1}{2}Y - \frac{1}{2}Z \\ x = -\frac{1}{2}X + \frac{1}{2}Y + \frac{1}{2}Z . \end{cases}$

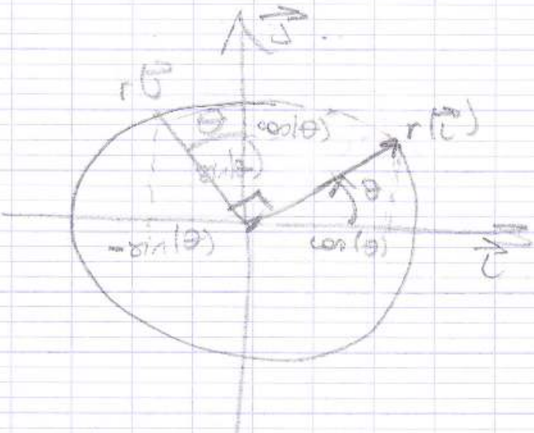
$$A^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Exercice n° 7: $E = \mathbb{R}^2$

$\mathcal{B} = (\vec{i}, \vec{j})$

$r . \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotation d'angle θ

$$\mathrm{Mat}_{\mathcal{B}}(r) = \begin{array}{cc} \begin{array}{cc} r(\vec{i}) & r(\vec{j}) \end{array} & \\ \begin{pmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos(\theta) \end{pmatrix} & \begin{array}{c} \vec{i} \\ \vec{j} \end{array} \end{array}$$



$r(\vec{i}) = \cos(\theta)\vec{i} + \sin(\theta)\vec{j}$
$r(\vec{j}) = -\sin(\theta)\vec{i} + \cos(\theta)\vec{i}$

Cas général: $r\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$.

$\begin{pmatrix} x \\ y \end{pmatrix} = x\vec{i} + y\vec{j}$

r linéaire $\Rightarrow r\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x\, r(\vec{i}) + y\, r(\vec{j})$

$= x\cos(\theta)\vec{i} + x\sin(\theta)\vec{j} - y\sin(\theta)\vec{i} + y\cos(\theta)\vec{j}$

$= \begin{pmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{pmatrix}$.

Autre methode:

$$r\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta + \cos\theta \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + \cos\theta y \end{pmatrix}.$$

2) $E = \mathbb{R}^3$

$\mathcal{B} = (\vec{i}, \vec{j}, \vec{k})$

$r . \mathbb{R}^3 \to \mathbb{R}^3$ rotation d'axe (O_3) d'angle θ

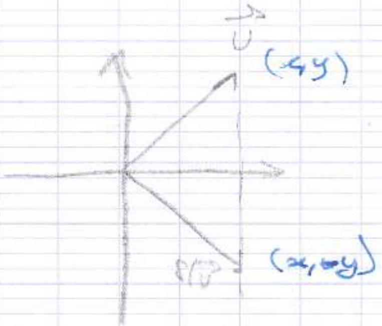
$$\mathrm{Mat}_{\mathcal{B}}(r) = \begin{array}{c} \begin{array}{ccc} r(\vec{i}) & r(\vec{j}) & r(\vec{k}) \end{array} \\ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \vec{i} \\ \vec{j} \\ \vec{k} \end{array}$$

3) $E = \mathbb{R}^2$

$s =$ sym $\perp$ /(Ox)

$\mathcal{B} = (\vec{i}, \vec{j})$

$$\mathrm{Mat}_{\mathcal{B}}(s) = \begin{array}{c} \begin{array}{cc} s(\vec{i}) & s(\vec{j}) \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ \vec{i} \\ \vec{j} \end{array}$$



Exercice n°12 : Ⓗ $\begin{cases} \dim E < +\infty \\ (u,v) \in \mathcal{L}^2(E) \end{cases}$

1) * Ⓗ u injective AD) u surjective

$\Rightarrow \ker(u) = \{0_E\}$ $\qquad \Rightarrow \mathrm{Im}(u) = E$

Par théorème du Rang, on a :

$\dim E = \dim(\ker(u)) + \dim(\mathrm{Im}(u))$.

$\dim E = 0 + \dim(\mathrm{Im}(u))$

$\dim E = \dim(\mathrm{Im}(u))$.

Or $\mathrm{Im}(u) \subset E$. Donc $E = \mathrm{Im}(u)$.

* Ⓗ u surjective AD) u injective

Par le théorème du Rang

$\dim E = \dim(\ker(u)) + \dim(\mathrm{Im}(u))$

$\dim(\ker(u)) = \dim(E) - \dim(\mathrm{Im}(u))$

Or $\mathrm{Im}(u) \subset E$. D'où $\ker(u) = \{0_E\}$.

Alors $\dim(E) - \dim(\mathrm{Im}(u)) = 0$.

Maths

18/04

Methodo

2) Ⓗ $u \circ v = id$ AD] u surjective

$Im(u) = E$

$A \subset B$ code la route

Soit x dans A.

$\boxed{\supset}$ Ok par définition

$\boxed{\subset}$ Soit x dans E.

Par Ⓗ $x = u \circ v(x)$

$\Rightarrow x = u(v(x))$

$\Rightarrow x \in Im(u)$

3) Ⓗ $v \circ u = id$ AD] u injective $Ker(u) = \{0_E\}$.

$\boxed{\supset}$ Ok par défi.

$\boxed{\subset}$ Soit $x \in Ker(u)$.

D'où $u(x) = 0_E$

$\Rightarrow v(u(x)) = v(0_E)$.

$\overset{Ⓗ}{\Rightarrow}$ $x = v(0_E)$.

$\Rightarrow x = 0_E$ car v linéaire.

4) Ⓗ $(A, B) \in \mathcal{M}_n^2(\mathbb{R})$ tq $AB = I_n$.

AD1] A inversible

Soit $\mathcal{B}$ la base canonique de $\mathbb{R}^n$

Soient $(u, v) \in \mathcal{L}^2(\mathbb{R}^n)$ tq

$A = Mat_{\mathcal{B}}(u)$ et $B = Mat_{\mathcal{B}}(v)$

$AB = I_n \Leftrightarrow Mat_{\mathcal{B}}(u) \times Mat_{\mathcal{B}}(v) = Mat_{\mathcal{B}}(id)$

$\Leftrightarrow Mat_{\mathcal{B}}(u \circ v) = Mat_{\mathcal{B}}(id)$

D'où $u \circ v = id$

Ainsi par 2°), u est surjective.

Donc par 1°), u est bijective

CCL: A inversible.

AD2] $BA = I_n$

$AB = I_n$

$\Rightarrow A^{-1}(AB) = A^{-1} I_n$

$\Rightarrow (A^{-1}A)B = A^{-1}$

$\Rightarrow I_n B = A^{-1}$

$\Rightarrow B = A^{-1}$

$\Rightarrow BA = A^{-1}A$

$\Rightarrow BA = I_n$.

Exercice n°13:

1) $J^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$J^2 - J - 2I = 0$

$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 0$

$J^2 - J - 2I = 0$

$\Rightarrow J^2 - J = 2I$

$\Rightarrow J(J - I) = 2I$

$\Rightarrow J \times \frac{1}{2}(J - I) = I$

Donc J est inversible et

$J^{-1} = \frac{1}{2}(J - I) = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

2) Rappel : $A = BQ + R$

$d°(R) < d°(B)$.

Par div euclidienne:

$\exists (Q, R) \in \mathbb{R}[x]^2$

$X^n = (X^2 - X - 2)Q + R$ et $d°(R) < 2$

$= (X^2 - X - 2)Q + aX + b$ avec $(a, b) \in \mathbb{R}$.

$\Rightarrow X^n = (X+1)(X-2)Q(X) + aX + b$.

Ainsi pour $X = -1$ on a $(-1)^n = -a + b$ (1)

et pour $X = 2$ on a $2^n = 2a + b$ (2)

(2) - (1) $3a = 2^n - (-1)^n$

$\Rightarrow a = \frac{2^n - (-1)^n}{3}$

Ainsi $b = (-1)^n + a = \frac{3(-1)^n + 2^n - (-1)^n}{3} = \frac{2(-1)^n + 2^n}{3}$

3) Via 2°)

$J^n = \underbrace{(J^2 - J - 2I)}_{=0} Q(J) + aJ + bI$

$J^n = aJ + bI = \frac{2^n - (-1)^n}{3} J + \frac{2(-1)^n + 2^n}{3} I$.