TD : Fractions rationnelles

14/03

$$F_1 = \frac{X^4+50}{(X-2)(X-3)} = E + \frac{a}{X-2} + \frac{b}{X-3}$$

$d°(F_1) = 4-2 \geqslant 0$

$$F_2 = \frac{X-1}{(X-2)^3(X-3)^2} = \underset{0}{\underset{\shortparallel}{E}} + \frac{a}{X-2} + \frac{b}{(X-2)^2} + \frac{c}{(X-3)} + \frac{d}{(X-3)^2} + \frac{e}{(X-3)^3}$$

$d°(F_2) = 1-5 = -4 < 0$

$$F_3 = \frac{X^6+1}{(X-1)(X-2)^2(X^2+X+1)} = E + \ldots$$

$$F = \frac{P}{Q} = \frac{EQ+R}{Q} = E + \frac{R}{Q}$$

$A = BQ + R \quad d°(R) < d°(Q)$

$F = \underset{\text{poly}}{E} + \underset{\text{fraction}}{F_1}$

$d°(F_1) < 0$

o) $F = \dfrac{X^4+1}{(X-1)(X+1)}$

$d°(F) = 4-2 \geqslant 0 \rightsquigarrow$ Calcul de E et F_1

$$\begin{array}{r|l} X^4 \quad +1 & X^2+1 \\ -X^4 + X^2 \quad & X^2-1 \\ \hline -X^2+1 & \\ X^2+1 & \\ \hline 2 & \end{array}$$

$$F = \frac{(X-1)(X+1)(X^2+1)+2}{(X-1)(X+1)} = \underbrace{X^2+1}_{E} + \underbrace{\frac{2}{(X-1)(X+1)}}_{F_1}$$

* $F_1 = \dfrac{2}{(X-1)(X+1)} = \dfrac{a}{(X-1)} + \dfrac{b}{(X+1)}$

- $(X-1)F_1 = \dfrac{2}{X+1} = a + \dfrac{b(X-1)}{X+1} \Rightarrow [(X-1)F_1]_{X=1} = 1 = a$
- $[(X+1)F_1]_{X=-1} = -1 = b$

CCL : $F = (X^2+1) + \dfrac{1}{X-1} + \dfrac{-1}{X+1}$

Exercice n°1 :

1) $\frac{x^3-1}{(x-1)(x-2)(x-3)} = F = \frac{(x-1)(\quad ? \quad)}{(x-1)(x-2)(x-3)}$

Simplification de F :

$$\begin{array}{r|l} x^3 \quad\quad -1 & x-1 \\ -\ x^3 - x^2 \quad\quad & x^2+x+1 \\ \hline x^2 \quad -1 & \\ -\ x^2 - x \quad & \\ \hline x-1 & \\ -x+1 & \\ \hline 0 & \end{array}$$

$F = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$

degré de F : $d°(F) = 2-2 = 0$ ⤳ Calcul de E et F_1

Développement du dénominateur : x^2-5x+6.

$$\begin{array}{r|l} x^2+x+1 & x^2-5x+6 \\ -\ x^2+5x-6 & 1 \\ \hline +6x-5 & \end{array}$$

D'où $F = \frac{(x-2)(x-3)*1+(6x-5)}{(x-2)(x-3)} = \underbrace{1}_{E} + \underbrace{\frac{6x-5}{(x-2)(x-3)}}_{F_1}$

* $F_1 = \frac{6x-5}{(x-2)(x-3)} = \frac{a}{x-2} + \frac{b}{x-3}$

$(x-2)F_1 =$

- $[(x-2)F_1]_{x=2} = \frac{6\times2-5}{2-3} = a \Rightarrow a = -7$
- $[(x-3)F_1]_{x=3} = \frac{6\times3-5}{3-2} = b \Rightarrow b = 13$

CCL : $F = 1 - \frac{7}{x-2} + \frac{13}{x-3}$

1,5) $F = \frac{*}{(x-2)(x-1)(x+1)^2}$

$d°(F) = 1-4 = -3 < 0$

$\Rightarrow E = 0$ et $F_1 = F$

$$F = \frac{X}{(X-2)(X-1)(X+1)^2} = \frac{a}{(X-2)} + \frac{b}{X-1} + \frac{c}{X+1} + \frac{d}{(X+1)^2}$$

$* \left[(X+1)^2 F\right]_{X=-1} = \frac{-1}{-3 \times (-2)} = d$ d'où $d = \frac{-1}{6}$

$* \left[(X-2) F\right]_{X=2} = \frac{2}{1 \times 3^2} = a \Rightarrow a = \frac{2}{9}$

$* \left[(X-1) F\right]_{(X=1)} = \frac{1}{-1 \times 4} = b \Rightarrow b = \frac{-1}{4}$

Pour c :

1e methode :

$F(0) = 0 = -\frac{a}{2} - b + c + d \Rightarrow c = \frac{a}{2} + b - d = \frac{1}{9} - \frac{1}{4} + \frac{1}{6} = \frac{1}{36}$

2e methode :

$\lim_{X \to +\infty} XF = 0 = a + b + c$

$\Rightarrow c = -a - b = \frac{-2}{9} + \frac{1}{4} = \frac{1}{36}$

CCL : $F = \frac{\frac{2}{9}}{X-2} - \frac{\frac{1}{4}}{X-1} + \frac{1/36}{X+1} - \frac{\frac{1}{6}}{(X+1)^2}$

2) $F = \frac{X^4 + X^2 + 2}{(X+1)(X+2)^2}$

Developpement du dénominateur : $(X+1)(X^2 + 4X + 4) = (X^3 + 4X^2 + 4X + X^2 + 4X + 4)$
$= X^3 + 5X^2 + 8X + 4$

$$\begin{array}{r|l} X^4 + X^2 + 2 & X^3 + 5X^2 + 8X + 4 \\ \hline - \cancel{X^4} - 5X^3 - 8X^2 - 4X & X - 5 \\ \hline -\cancel{5X^3} - 7X^2 - 4X + 2 & \\ \cancel{5X^3} + 25X^2 + 40X + 20 & \\ \hline 18X^2 + 36X + 22 & \end{array}$$

$$F = X - 5 + \frac{18X^2 + 36X + 22}{(X+1)(X+2)^2}$$

$* F_1 = \frac{18X^2 + 36X + 22}{(X+1)(X+2)^2} = \frac{a}{X+1} + \frac{b}{X+2} + \frac{c}{(X+2)^2}$

$\left[(X+1) F_1\right]_{X=-1} = \frac{18 - 36 + 22}{(-1+2)^2} = a \Rightarrow a = 4$

$$[(x+2)^2 F_1]_{x=-2} = \frac{4\times 18 - 2\times 36 + 22}{-1} = c \Rightarrow c = -22$$

$$\cancel{[F_1]_{x=0} = 0 = a + \frac{b}{2} + \frac{c}{4} \Rightarrow b = 8\left(a + \frac{c}{4}\right)}$$

$$\cancel{b = 2a + \frac{c}{2} = 8 - 11 = -3}$$

$$\lim_{x \to +\infty} xF_1 = 18 = a + b \Rightarrow b = 18 - a = 14$$

CCL : $F = x - 5 + \frac{4}{x+1} + \frac{14}{x-2} + \frac{-22}{(x-2)^2}$

3) $F = \frac{x^2-4}{(x-1)^2(x+1)^2} = \frac{P(x)}{Q(x)}$ $\quad d°(P) < d°(Q) \Rightarrow E(x) = 0$

$$F = \frac{x^2-4}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{(x+1)} + \frac{d}{(x+1)^2}$$

$(x^2+x+1)(x^2+x+1)$

$$[(x-1)^2 F]_{x=1} = \frac{-3}{4} = b$$

$$[(x+1)^2 F]_{x=-1} = \frac{-3}{4} = d.$$

$x^4 + x^3 + x^2$

$$\lim_{x \to +\infty} xF = 0 = a + 0 + c + 0$$

$\Rightarrow c = -a$

$$F(0) = -4 = -a + b + c + d = -a + b - a + d = -2a + b + d$$

$\Rightarrow -2a = -4 - b - d = -4 + \frac{3}{2} = \frac{-5}{2}$

$\Rightarrow a = \frac{5}{4}$

D'où $c = \frac{-5}{4}$

CCL : $F = \frac{\frac{5}{4}}{x-1} + \frac{\frac{-3}{4}}{(x-1)^2} + \frac{\frac{-5}{4}}{(x+1)} + \frac{\frac{-3}{4}}{(x+1)^2}$

Maths
17/05
2

5) $F = \frac{2X^4+1}{(X-1)^3(X^2+1)}$ $\deg(2X^4+1) < \deg((X-1)^3(X^2+1))$.
$E(X)=0$

$$F = \frac{2X^4+1}{(X-1)^3(X^2+1)} = \frac{a}{X-1} + \frac{b}{(X-1)^2} + \frac{c}{(X-1)^3} + \frac{dX+e}{X^2+1}$$

$$[(X-1)^3F]_{X=1} = \frac{2+1}{2} = c \Rightarrow c = \frac{3}{2}$$

$$[(X^2+1)F]_{X=i} = \frac{2+1}{(i-1)^3} = di+e \Leftrightarrow \frac{2+1}{2i+2} = di+e$$

$$= \frac{3\times(-2i+2)}{(2i+2)(-2i+2)} = di+e$$

$$= \frac{-6i+6}{4+4} = di+e \Leftrightarrow \frac{-6i+6}{8} = di+e$$

$(i-1)(i-1)(i-1)$
$\cancel{-1}-i-i+\cancel{1}$
$-2i$
$+2+2i$

$$d = \frac{-6}{8} \text{ et } e = \frac{6}{8}$$
$$d = \frac{-3}{4} \quad e = \frac{3}{4}$$

$$\lim_{x\to+\infty} xF = 2 = a+d$$
$$\Rightarrow a = 2-d = 2+\frac{3}{4} = \frac{11}{4}$$

$$F(0) = -1 = -a+b-c+e$$
$$\Rightarrow b = -1+a+c-e$$
$$= -1+\frac{11}{4}+\frac{3}{2}-\frac{3}{4} = \frac{10}{4} = \frac{5}{2}.$$

CCL : $F = \frac{\frac{11}{4}}{X-1} + \frac{\frac{5}{2}}{(X-1)^2} + \frac{\frac{3}{2}}{(X-1)^3} + \frac{-\frac{3}{4}X+\frac{3}{4}}{X^2+1}$

Exercice n° 2 :

1) $\int_0^1 \frac{x^3+x^2+2}{(x+1)(x^2+2)} dx$

$F = \frac{x^3+x^2+2}{(x+1)(x^2+1)}$ deg (num) = deg (denum)

Partie entière :

Denominateur : x^3+x^2+x+1

$$\begin{array}{l|l} x^3+x^2+2 & x^3+x^2+x+1 \\ -x^3-x^2-x-1 & 1 \\ \hline -x+1 & \end{array}$$

$F = 1 + \frac{-x+1}{(x+1)(x^2+1)}$

$F_1 = \frac{-x+1}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$

$[(x^2+1)F_1]_{x=i} = \frac{-i+1}{i+1} = bi+c$

$= \frac{(-i+1)(i+1)}{(i+1)(-i+1)} = bi+c =$

$= \frac{\cancel{-1}-2i+\cancel{1}}{1+1} = bi+c \iff -i+0 = bi+c$

$\Rightarrow b = -1 \quad c = 0$.

$\lim_{x\to+\infty} xF_1 = 0 = a+b$

$\Rightarrow a = -b \iff a = 1$ ✓

Donc $F = 1 + \frac{1}{x+1} - \frac{x}{x^2+1}$

$\int_0^1 f(x)dx = \left[x + \ln(x+1) - \frac{1}{2}\ln(x^2+1)\right]_0^1$

$\int_0^1 f(x)dx = 1 + \ln(2) - \frac{1}{2}\ln(2) = 1 + \frac{1}{2}\ln(2)$

2) $f(x) = \dfrac{x^4+1}{(x+1)^2(x-2)}$

Partie entière : $(x^2+2x+1)(x-2) = x^3-3x-2$

$$\begin{array}{r|l} x^4+1 & x^3-3x-2 \\ \hline 3x^2+2x+1 & x \end{array}$$

$a = \dfrac{10}{9}$

$b = \dfrac{-2}{3}$

$c = \dfrac{17}{9}$

$$f(x) = x + \frac{3x^2+2x+1}{(x+1)^2(x-2)}$$

$$f_1(x) = \frac{3x^2+2x+1}{(x+1)^2(x-2)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x-2}$$

Algèbre linéaire 1 et 2 + Fractions Rationnelles

$$\int_0^1 f(x) = \int_0^1 x + \frac{10}{9(x+1)} - \frac{2}{3(x+1)^2} + \frac{17}{9(x-2)}\,dx$$

$$= \left[\frac{x^2}{2} + \frac{10}{9}\ln(|x+1|) + \frac{2}{3(x+1)} + \frac{17}{9}\ln(|x-2|)\right]_0^1$$

$$= \frac{1}{2} + \frac{10}{9}\ln(2) + \frac{1}{3} - \frac{2}{3} - \frac{17}{9}\ln(2) =$$

$$= \frac{-7}{9}\ln(2) + \frac{1}{6}$$