TD Test 4

Name : **First name :** **Class :**

Question from the class

Let $\sum f_n$ be a series of functions converging pointwise on an interval I of $\mathbb{R}$.

1. Give the definition of the sequence of the remainder functions (R_n).

$$R_n(x) = \sum_{k=n+1}^{+\infty} f_k(x)$$

2. Give a necessary and sufficient condition for $\sum f_n$ to converge uniformly on I.

$\sum f_n$ converges uniformly $\iff$ $(R_n) \xrightarrow{CVU}$ null function

3. Give the definition of the normal convergence of $\sum f_n$ on I.

$\sum \sup |f_n|$ is CV.

Exercise 1

Let (f_n) be the sequence of functions defined for every $x \in [0,1]$ by $f_n(x) = \dfrac{nx}{nx+1}$.

1. Study the pointwise and uniform convergence of (f_n) on $[0,1]$.

* pointwise CV: let $x \in [0,1]$
  - If $x=0$, then $f_n(x) = 0 \xrightarrow[n\to+\infty]{} 0$
  - Else, $f_n(x) \sim \frac{nx}{nx}$ so $\lim f_n(x) = 1$

Finally, $(f_n) \xrightarrow{\text{pointwise}} f$ with $f(x) = \begin{cases} 0 & \text{if } x=0 \\ 1 & \text{if } x \neq 0 \end{cases}$

* CVU:

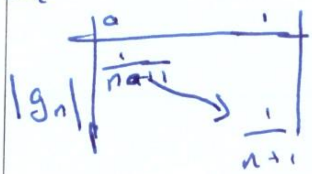
$\forall n \in \mathbb{N}$, f_n is continuous. Yet, f is not continuous.

So (f_n) does not converge uniformly to f

2. Let $a \in]0,1[$. Study the uniform convergence of (f_n) on $[a,1]$.

Let $n \in \mathbb{N}$ and $g_n(x) = f_n(x) - f(x) = f_n(x) - 1$

Then $\forall x \in [0,1]$, $g_n(x) = f_n(x) - 1 = -\dfrac{1}{nx+1} \Rightarrow |g_n(x)| = \dfrac{1}{nx+1}$



$|g_n|$ is $\searrow$ on $[0,1]$ and

$$\sup_{[0,1]} |g_n| = |g_n(a)| = \frac{1}{na+1}$$

Since $\dfrac{1}{na+1} \to 0$, $(f_n) \xrightarrow{CVU} f$

Exercise 2

Let (f_n) be the sequence of functions defined for every $x \in [0,1]$ by $f_n(x) = \dfrac{ne^{-x} + x^2}{n+x}$. Study the pointwise and uniform convergence of (f_n) on $[0,1]$.

* Pointwise CV: let $x \in [0,1]$.

Then $f_n(x) = \dfrac{ne^{-x} + x^2}{n+x} = \dfrac{\cancel{n}\left(e^{-x} + \frac{x^2}{n}\right)}{\cancel{n}\left(1 + \frac{x}{n}\right)} \longrightarrow e^{-x}$

So $(f_n) \xrightarrow{\text{pointwise}} f$ with $f : x \longmapsto e^{-x}$

* Uniform CV: let $n \in \mathbb{N}$ and $g_n = f_n - f$

Then $\forall x \in [0,1]$, $g_n(x) = \dfrac{ne^{-x} + x^2}{n+x} - e^{-x} = \dfrac{x^2 - xe^{-x}}{n+x}$

- The numerator does not depend on n and is bounded on $[0,1]$: $\forall x \in [0,1]$, $x^2 - xe^{-x} = x(x - e^{-x})$ and

$$\left.\begin{array}{l} 0 \leq x \leq 1 \\ -1 \leq -e^{-x} \leq x - e^{-x} \leq x \leq 1 \end{array}\right\} \Rightarrow |x(x-e^{-x})| \leq 1$$

- So $\forall x \in [0,1]$, $|g_n(x)| = \dfrac{|x(x-e^{-x})|}{n+x} \leq \dfrac{1}{n+x} \leq \dfrac{1}{n}$

does not depend on x

Since $\frac{1}{n} \to 0$, $(f_n) \xrightarrow{CVU} f$