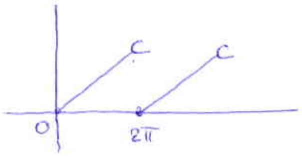
5)



$a_0(f)=\frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}[x^2]_0^{2\pi} = 2\pi$

$a_n(f)=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_0^{2\pi} x\cos(nx)dx$

ni paire, ni impaire

$u(x)=x \Rightarrow u'(x)=+1;\ v(x)=\frac{1}{n}\sin(nx) \Leftarrow v'(x)=\cos(nx)$

$a_n(f)=\frac{1}{\pi}\left(\frac{1}{n}\underbrace{[x\sin(nx)]_0^{2\pi}}_{=0} - \frac{1}{n}\underbrace{\int_0^{2\pi}\sin(nx)dx}_{=0}\right)=0$

$b_n(f)=\frac{1}{\pi}\int_0^{2\pi}\underbrace{f(x)}_{=x}\sin(nx)dx$

$u(x)=x\Rightarrow u'(x)=1;\ v(x)=\frac{-1}{n}\cos(nx)\Leftarrow v'(x)=\sin(nx)$

$b_n(f)=\frac{1}{\pi}\left(\frac{-1}{n}[x\cos(nx)]_0^{2\pi}+\frac{1}{n}\underbrace{\int_0^{2\pi}\cos(nx)dx}_{=0}\right) = \frac{-1}{n\cancel{\pi}}\cdot 2\cancel{\pi} = \frac{-2}{n}$

la série de Fourier de f est: $\underbrace{\pi}_{\frac{a_0(f)}{2}} - 2\sum\frac{1}{n}\sin(nx)$. f est C^1 par morceaux et C^0 sur $]0, 2\pi[$.

donc $\underbrace{f(x)}_{=x}=\pi-2\sum_{n=1}^{+\infty}\frac{1}{n}\sin(nx)$

En ~~x=~~ $\frac{\pi}{2}$: $\frac{\pi}{2}=\pi-2\sum_{\substack{\text{par}\\ k=0}}^{+\infty}\frac{1}{2k+1}\overbrace{\sin\left((2k+1)\frac{\pi}{2}\right)}^{=(-1)^k} \Rightarrow \boxed{\sum_{k=0}^{+\infty}\frac{(-1)^k}{2k+1}=\frac12\left(\pi-\frac{\pi}{2}\right)=\frac{\pi}{4}}$

Via Parseval,

$\frac{a_0^2(f)}{4}+\frac12\sum\left(a_n^2(f)+b_n^2(f)\right) = \frac{1}{2\pi}\int_0^{2\pi}f^2(x)dx = \frac{1}{2\pi}\int_0^{2\pi}x^2dx = \frac{1}{2\pi}\cdot\frac13[x^3]_0^{2\pi}=\frac{4\pi^2}{3}$

$\underbrace{\pi^2}_{a_0^2}+2\sum_{n=1}^{+\infty}\underbrace{\frac{1}{n^2}}_{b_n^2}=\frac{4\pi^2}{3} \Rightarrow \sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac12\left(\frac{4\pi^2}{3}-\pi^2\right)=\frac{\pi^2}{6}$

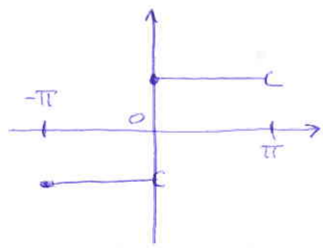
Exercice n°20 :

1)

$a_0(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=0$

$a_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx=0$

$b_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx=\frac{2}{\pi}\int_0^{\pi}\sin(nx)dx = \frac{-2}{n\pi}[\cos(nx)]_0^{\pi}$



$b_n(f)=\frac{-2}{n\pi}((-1)^n-1)$.

La série de Fourier de f est: $\frac{-2}{\pi}\sum\frac1n((-1)^n-1)\sin(nx)$

$b_{2p}(f)=0;\ b_{2p+1}(f)=\frac{4}{(2p+1)\pi}$

La série de Fourier peut également s'écrire : $\frac4\pi\sum\sin((2p+1)x)$

2) Comme f est C^1 par morceaux et continue sur $]0,\pi[$, $\forall x\in]0,\pi[$, $\overbrace{f(x)}^{=1}=\frac4\pi\sum_{p=0}^{+\infty}\frac{1}{2p+1}\sin((2p+1)x)$

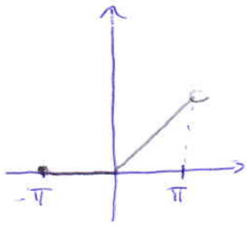
En particulier pour $x = \frac{\pi}{2}$, $1 = \frac{4}{\pi}\sum_{p=0}^{+\infty}\frac{(-1)^p}{2p+1} \Rightarrow \sum_{p=0}^{+\infty}\frac{(-1)^p}{2p+1} = \frac{\pi}{4}$

Via parseval,

$$\frac{8}{\pi^2}\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{\pi}\int_0^{\pi} dx = 1.$$

$$\Rightarrow \boxed{\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2} = \frac{\pi^2}{8}}$$

Exercice n°21:



$$a_0(f) = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}\underbrace{f(x)dx}_{=0} + \int_0^{\pi} f(x)dx\right)$$

$$a_0(f) = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}.$$

$$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$u(x) = x \Rightarrow u'(x) = 1$; $v(x) = \frac{1}{n}\sin(nx) \Leftarrow v'(x) = \cos(nx)$

$$a_n(f) = \frac{1}{\pi}\left(\frac{1}{n}\underbrace{[x\sin(nx)]_0^{\pi}}_{=0} - \frac{1}{n}\int_0^{\pi}\sin(nx)dx\right) = \frac{1}{n\pi}\int_0^{\pi}\sin(nx)dx$$

$$a_n(f) = \frac{1}{n^2\pi}[\cos(nx)]_0^{\pi} = \frac{1}{n^2\pi}\left((-1)^n - 1\right)$$

$$b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)dx$$

$u(x) = x \Rightarrow u'(x) = 1$; $v(x) = \frac{-1}{n}\cos(nx) \Leftarrow v'(x) = \sin(nx)$.

$$b_n(f) = \frac{1}{\pi}\left(\frac{-1}{n}[x\cos(nx)]_0^{\pi} + \frac{1}{n}\underbrace{\int_0^{\pi}\cos(nx)dx}_{=0}\right) = -\frac{1}{n\pi}\cdot\pi\cdot(-1)^n = \frac{(-1)^{n+1}}{n}.$$

La série de Fourier de f est : $\frac{\pi}{4} + \sum\left(\frac{1}{n^2\pi}((-1)^n - 1)\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right)$

2) f est C^1 par morceaux et continue sur $]-\pi,\pi[$. $\forall x \in]-\pi,\pi[$

$$\frac{\pi}{4} + \sum\left(\frac{1}{n^2\pi}((-1)^n - 1)\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right)$$

En particulier pour $x = 0$.

$$0 = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n - 1}{n^2} = \frac{\pi}{4} - \frac{2}{\pi}\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2} \text{ donc } \sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2} = \frac{\pi^2}{8}$$