Fonctions à deux variables
TD.

Exercice n°1 :

1) $f(x,y) = x^2 + (x+y-1)^2 + y^2$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Longleftrightarrow \begin{cases} 2x + 2(x+y-1) = 0 & L_1 \\ 2(x+y-1) + 2y = 0 & L_2 \end{cases}$$

$L_2 \leftarrow L_2 - L_1 : x = y$.

donc via L_1, $2x + 2(2x-1) = 0 \Leftrightarrow 6x = 2 \Leftrightarrow x = \frac{1}{3}$

Un point critique : $\left(\frac{1}{3}, \frac{1}{3}\right)$

$\frac{\partial^2 f}{\partial x^2}(x,y) = 2+2 = 4$; $\frac{\partial^2 f}{\partial y^2}(x,y) = 4$; $\frac{\partial^2 f}{\partial x \partial y} = 2$.

En $\left(\frac{1}{3}, \frac{1}{3}\right)$: $r = 4$, $s = 2$ et $t = 4$.

$rt - s^2 = 16 - 4 = 12 > 0$; $r > 0$ donc $\left(\frac{1}{3}, \frac{1}{3}\right)$ est un minimum local.

2) $g(x,y) = x^3 + y^3 - 9xy + 1$

$$\begin{cases} \frac{\partial g}{\partial x}(x,y) = 0 \\ \frac{\partial g}{\partial y}(x,y) = 0 \end{cases} \Longleftrightarrow \begin{cases} 3x^2 - 9y = 0 \\ 3y^2 - 9x = 0 \end{cases}$$

$\Rightarrow L_1 : 9y = 3x^2$

$\Rightarrow y = \frac{1}{3}x^2 \underset{L_2}{\Rightarrow} 3 \cdot \frac{1}{9}x^4 - 9x = 0$

$\frac{x^4}{3} - 9x = 0$

$\Rightarrow x^4 - 27x = 0$

$\Rightarrow x(x^3 - 27) = 0$

$\Leftrightarrow \begin{cases} x = 0 \underset{L_1}{\Rightarrow} y = 0 \\ \text{ou} \\ x = 3 \underset{L_1}{\Rightarrow} y = 3 \end{cases}$

Deux points critiques : $(0,0)$ et $(3,3)$

$\frac{\partial^2 g}{\partial x^2}(x,y) = 6x$

$\frac{\partial^2 g}{\partial y \partial x}(x,y) = -9$

$\frac{\partial g^2}{\partial y^2}$

Au point $(0,0)$

$r = 0$; $s = -9$ et $t = 0$

$rt - s^2 = -81 < 0 \Rightarrow (0,0)$ est un point-col.

Au point $(3,3)$

$r = 18$, $s = -9$ et $t = 18$

$rt - s^2 = 243 > 0$; $r > 0$ donc $(3,3)$ est un minimum local

3) $h(x,y) = x^2 + y^3 - 2xy - y$

$$\begin{cases} \frac{\partial h}{\partial x}(x,y) = 0 \\ \frac{\partial h}{\partial y}(x,y) = 0 \end{cases} \Longleftrightarrow \begin{cases} 2x - 2y = 0 \\ 3y^2 - 2x - 1 = 0 \end{cases}$$

$L_1 \Rightarrow x = y \Rightarrow 3x^2 - 2x - 1 = 0 \Rightarrow x = 1$ ou $x = \frac{-1}{3}$

Deux points critiques : $(1,1)$ et $\left(\frac{-1}{3}, \frac{-1}{3}\right)$

$\frac{\partial^2 h}{\partial x^2}(x,y) = 2$; $\frac{\partial^2 h}{\partial x \partial y}(x,y) = -2$; $\frac{\partial^2 h}{\partial y^2} = 6y$

Point $(1,1)$

$r = 2$; $s = -2$ et $t = 6$

$\Rightarrow rt - s^2 = 8 > 0 \,/\, r > 0$

$\Rightarrow (1,1)$ minimum local

Point $\left(\frac{-1}{3}, \frac{-1}{3}\right)$

$r = 2$, $s = -2$ et $t = 6$

$rt - s^2 = -8 <$

$\Rightarrow \left(\frac{-1}{3}, \frac{-1}{3}\right)$ point-col.

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4) $\begin{cases} \dfrac{\partial i}{\partial x}(x,y)=0 \\ \dfrac{\partial i}{\partial y}(x,y)=0 \end{cases} \iff \begin{cases} 2x+y+2=0 \\ x+2y+3=0 \end{cases}$ $L_2 \leftarrow L_2 - 2L_1$

$-3x-1=0 \Rightarrow x = \dfrac{-1}{3} \underset{L_1}{\Rightarrow} y = \dfrac{-4}{3}$

Un point critique: $\dfrac{\partial^2 i}{\partial x^2}(x,y)=2$; $\dfrac{\partial^2 i}{\partial x \partial y}(x,y)=1$; $\dfrac{\partial^2 i}{\partial y^2}(x,y)=2$

Au point $\left(\frac{-1}{3}, \frac{-4}{3}\right)$, $r=2$; $s=1$; $t=2$; $rt-s^2=3>0$ et $r>2$

$\Rightarrow \left(\frac{-1}{3}, \frac{-4}{3}\right)$ est un minimum local.

5) $\begin{cases} \dfrac{\partial j}{\partial x}(x,y)=0 \\ \dfrac{\partial j}{\partial y}(x,y)=0 \end{cases} \iff \begin{cases} \ln^2(x)+y^2+x.\,2\ln(x).\,\dfrac{1}{x}=0 \\ 2xy=0 \end{cases}$

$L_2 \Rightarrow y=0 \underset{L_1}{\Rightarrow} \ln(x)(\ln(x)+2)=0 \Rightarrow \begin{cases} \ln(x)=0 \\ \text{ou} \\ \ln(x)=-2 \end{cases} \Rightarrow \begin{cases} x=1 \\ \text{ou} \\ x=e^{-2} \end{cases}$

Deux points critiques: $(1,0)$ et $(e^{-2},0)$

$\dfrac{\partial j^2}{\partial x^2}(x,y) = \dfrac{2\ln(x)}{x} + \dfrac{2}{x} = \dfrac{2}{x}(\ln(x)+1)$

$\dfrac{\partial j^2}{\partial x \partial y}(x,y) = 2y$ $\quad \dfrac{\partial j^2}{\partial y^2} = 2x$

Au point $(1,0)$:

$r=2$; $s=0$ et $t=2$ $\quad rt-s^2=4>0$ et $r>0$ donc $(1,0)$ est un minimum local.

Au point $(e^{-2},0)$:

$r=-2e^{-2}$; $s=0$ et $t=2e^{-2}$ $\quad rt-s^2=-4<0$ donc $(e^{-2},0)$ est un point-col

6) $\begin{cases} \dfrac{\partial k}{\partial x}(x,y)=0 \\ \dfrac{\partial k}{\partial y}(x,y)=0 \end{cases} \iff \begin{cases} 6xy-6x=0 \\ 3x^2+3y^2-6y=0 \end{cases}$

$L_1 \Rightarrow x(y-1)=0 \Rightarrow x=0$ ou $y=1$

$x=0 \underset{L_2}{\Rightarrow} y(y-2)=0 \Rightarrow y=0$ ou $y=2$

$y=1 \underset{L_2}{\Rightarrow} 3x^2-3=0 \Rightarrow x=1$ ou $x=-1$

4 points critiques : $(0,0)$, $(0,2)$, $(1,1)$ et $(-1,1)$

$\dfrac{\partial^2 k}{\partial x^2}(x,y)=6y-6$ $\quad \dfrac{\partial^2 k}{\partial x \partial y}(x,y)=6x$ $\quad \dfrac{\partial^2 k}{\partial y^2}(x,y)=6y-6$

Au point $(0,0)$:
$r=t=-6$ et $s=0$
$rt-s^2=36>0$
$r<0$
$\Rightarrow (0,0)$ maximum local

Au point $(0,2)$:
$r=t=6$ et $s=0$
$rt-s^2=36>0$
$r>0$
$\Rightarrow (0,2)$ minimum local

Au point $(1,1)$:
$r=t=0$ et $s=6$
$rt-s^2=-36<0$
$\Rightarrow (1,1)$ point col

Au point $(-1,1)$:
$r=t=0$ et $s=-6$
$rt-s^2=-36<0$
$\Rightarrow (1,-1)$ est un point col

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7) $\begin{cases} \frac{\partial f}{\partial x}(x,y)=0 \\ \frac{\partial f}{\partial y}(x,y)=0 \end{cases} \Leftrightarrow \begin{cases} 6x^2-6x=0 \\ 6y^2-6y=0 \end{cases} \Leftrightarrow \begin{cases} x(x-1)=0 \\ y(y-1)=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \text{ ou } x=1 \\ y=0 \text{ ou } y=1 \end{cases}$

4 points critiques : $(0,0),(0,1),(1,0),(1,1)$

$\frac{\partial^2 f}{\partial x^2}(x,y)=12x-6$; $\frac{\partial^2 f}{\partial x \partial y}(x,y)=0$; $\frac{\partial^2 f}{\partial y^2}(x,y)=12y-6$

Au point $(0,0)$:
$r=-6$; $s=0$; $t=-6$
$rt-s^2=36>0$
$r<0$
$\Rightarrow (0,0)$ est un maximum local

Au point $(0,1)$:
$r=-6$; $s=0$; $t=6$
$rt-s^2=-36<0$
$\Rightarrow (0,1)$ est un point-col

Au point $(1,0)$
$r=6$; $s=0$; $t=-6$
$rt-s^2=-36<0$
$\Rightarrow (1,0)$ est un point-col

Au point $(1,1)$
$r=6$; $s=0$; $t=6$
$rt-s^2=36>0$
$r>0$
$\Rightarrow (1,1)$ est un minimum local

Exercice n°2 :

1) $\frac{\partial^2 f}{\partial x^2}(x,y)=0 \Rightarrow \frac{\partial f}{\partial x}(x,y)=g(y) \Rightarrow f(x,y)=x\,g(y)+h(y)$

Réciproquement, si $f(x,y)=x\,g(y)+h(y)$ alors $\frac{\partial^2 f}{\partial x^2}(x,y)=0$

2) $\frac{\partial^2 f}{\partial x \partial y}(x,y)=0 \Rightarrow \frac{\partial f}{\partial y}(x,y)=g(y) \Rightarrow f(x,y)=G(y)+H(x)$

Réciproquement, si $f(x,y)=G(y)+H(x)$

$\frac{\partial^2 f}{\partial x \partial y}(x,y)=0$