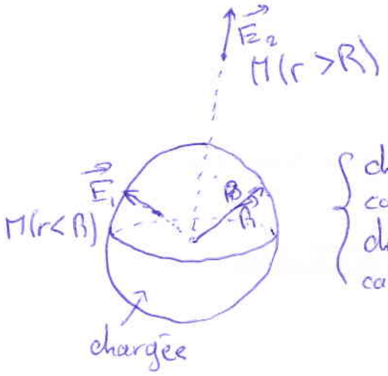
Exercice n°1 : Soit une distribution sphérique de charges de densité ρ

Cette distribution crée un $\vec{E}$ radial

$$E_r(r) = \frac{k}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^3}{5R^2}\right)$$

radiale ↗ ↑ dépend de r



$\left\{\begin{array}{l} \mathrm{div}(\vec{E}_2) = 0 \\ \text{car } \rho = 0 \\ \mathrm{div}(\vec{E}_1) > 0 \\ \text{car } \rho > 0 \end{array}\right.$

$$\boxed{\mathrm{div}(\vec{E}) = \frac{\rho}{\varepsilon}}$$

On donne $\mathrm{div}_{sphere}(\vec{E})$ pour un champ radial

$$\mathrm{div}(\vec{E}) = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2 E_r) + (\ )\underbrace{E_\theta}_{=0} + (\ )\underbrace{E_3}_{=0}$$

$$\mathrm{div}(\vec{E}) = \frac{\rho}{\varepsilon} = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2 E_r)$$

$$= \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{k}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^3}{5R^2}\right)\right) = \frac{1}{r^2}\cdot\frac{k}{\varepsilon_0}\cdot\frac{\partial}{\partial r}\left(\frac{r^3}{3} - \cancel{\phi}\,\frac{r^5}{5R^2}\right)$$

$$= \frac{1}{r^2}\cdot\frac{k}{\varepsilon_0}\left(r^2 - \frac{r^4}{R^2}\right) = \frac{k}{\varepsilon_0}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\Rightarrow \rho = k\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\mathrm{div}_{sphere}(\vec{E}) = \frac{1}{r^4}\cdot\frac{\partial}{\partial r}\left(r^2\cdot\frac{k}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^3}{5R^2}\right)\right) = \frac{k}{\varepsilon_0}\cdot\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(\frac{r^3}{3} - \frac{r^5}{5R^2}\right)$$

$$= \frac{k}{\varepsilon_0}\cdot\frac{1}{r^2}\left(\frac{\cancel{3}r^2}{\cancel{3}} - \frac{\cancel{5}r^4}{\cancel{5}R^2}\right) = \frac{k}{\varepsilon_0}\left(1 - \frac{r^2}{R^2}\right).$$

or $\mathrm{div}(\vec{E}) = \dfrac{\rho}{\varepsilon_0}$ (D'après la 1ère équation de Maxwell)

$\rho(r) = \varepsilon_0\,\mathrm{div}(\vec{E}) = k\left(1 - \dfrac{r^2}{R^2}\right) = \cancel{k \cdot 1}$ $\Rightarrow$ • $[k] = [\rho] = C.m^{-3}$

• $r = 0$ on a $\rho = k = \rho_0$.

$$\boxed{\rho(r) = \rho_0\left(1 - \frac{r^2}{R^2}\right)}$$

k : densité de charge au centre de la sphère.

Exercice n°2 : Différé

Exercice n°3 :

1) Retrouver l'équation propre de $\vec{E}$ (milieu quelconque)

$$\Delta\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \frac{1}{\varepsilon}\,\overrightarrow{\mathrm{grad}}(\rho) + \mu\frac{\partial\vec{J}}{\partial t}$$

$$\Delta\vec{U} = \overrightarrow{\mathrm{grad}}(\mathrm{div}(\vec{U})) - \overrightarrow{\mathrm{rot}}(\overrightarrow{\mathrm{rot}}(\vec{U})) \quad (\vec{U} = \vec{E})$$

2) Réécrire l'équation de propagation de $\vec{E}$ dans le milieu air, en déduire C ?

3) $\vec{E}(x,t) = E_0\cos(kx - \omega t)\vec{E}_3$ est de l'équation propre. En déduire une relation entre k, ω etc

1) $\vec{\Delta E} = \vec{grad}(div(\vec{E})) - \vec{rot}(\vec{rot}(\vec{E})) = \vec{grad}\left(\frac{\rho}{\varepsilon}\right) - \vec{rot}\left(-\frac{\partial \vec{B}}{\partial t}\right)$

$\vec{\Delta E} = \frac{1}{\varepsilon}\vec{grad}(\rho) + \frac{\partial}{\partial t}\vec{rot}(\vec{B}) = \frac{1}{\varepsilon}\vec{grad}(\rho) + \frac{\partial}{\partial t}\left(\mu\vec{J} + \mu\varepsilon\frac{\partial \vec{E}}{\partial t}\right)$

$$\vec{\Delta E} = \frac{1}{\varepsilon}\vec{grad}(\rho) + \mu\frac{\partial \vec{J}}{\partial t} + \mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\Delta E} - \mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\varepsilon}\vec{grad}(\rho) + \mu\frac{\partial \vec{J}}{\partial t}$$

2) Dans le milieu air : $\rho = 0$; $\vec{J} = \vec{0}$; $\mu = \mu_0$, $\varepsilon = \varepsilon_0$

$$\boxed{\vec{\Delta E} - \mu_0\varepsilon_0\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}}$$

$\mu_0\varepsilon_0$?

$$\mu_0\varepsilon_0 = \frac{\Delta E_x}{\frac{\partial^2 E_x}{\partial t^2}} \Rightarrow [\mu_0\varepsilon_0] = \frac{[\Delta]}{\left[\frac{\partial^2}{\partial^2 t}\right]} = \frac{m^{-2}}{s^{-2}} = \frac{1}{m^2.s^{-2}} = \frac{1}{(m.s^{-1})^2}.$$

CCL : $\mu_0\varepsilon_0$ est homogène à l'inverse du carré d'une vitesse.

$$\mu_0\varepsilon_0 = \frac{1}{c^2} \Rightarrow \vec{\Delta E} - \frac{1}{c^2}\cdot\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \quad (*)$$

3) $\vec{E}(x,t) = E_0 \cos(\underbrace{k_x x}_{\vec{k}.\vec{OM} \text{ où } \vec{k}\begin{pmatrix}k\\0\\0\end{pmatrix}} - \omega t)\,\vec{e_3}$

$$\vec{\Delta E} = \begin{pmatrix} \Delta E_x = 0 \\ \Delta E_y = 0 \\ \Delta E_3 \end{pmatrix} \begin{matrix} \text{car } E_x = 0\ (\vec{E} \text{ sin } \vec{O_3}) \\ \text{car } E_y = 0\ (\vec{E} \text{ sin } \vec{O_3}) \\ \ \end{matrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^2 E_3}{\partial x^2} + \underbrace{\frac{\partial^2 E_y}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 E_3}{\partial x^2}}_{=0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -k^2 E_0\cos(kx - \omega t) \end{pmatrix} = -k^2\vec{E}$$

$\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2\vec{E}$

Dans $(*)$ $-k^2\vec{E} + \frac{1}{c^2}\omega^2\vec{E} = \vec{0}$.

$$\underbrace{\vec{E}}_{\neq 0}\underbrace{\left(-k^2 + \frac{\omega^2}{c^2}\right)}_{=0} = \vec{0}$$

$= 0 \Rightarrow k = \frac{\omega}{c}$ $/\omega = kc|$

Exercice n°4 Soient les vecteurs $(\vec{E},\vec{B})$: $\vec{E}(y,t) = E_0 \cos(ky-\omega t)\vec{e_z}$.

$\vec{B}(y,t) = B_0 \cos(ky-\omega t)\vec{e_x}$

Montrer que $\vec{E}$ et $\vec{B}$ vérifient les 4 équations de Maxwell dans le milieu air (H) $(E_0 \overset{\text{opps}}{=} cB_0, \omega = kc)$

Equations de Maxwell dans le milieu air

$$\begin{cases} \rho = 0 & \text{(pas de charge)} \\ \vec{J} = \vec{0} & \text{(pas de courant)} \\ \varepsilon = \varepsilon_0 \varepsilon_r^{\,\approx 1} = \varepsilon_0 \\ \mu = \mu_0 \mu_r = \mu_0 \\ \mu_0 \varepsilon_0 = \frac{1}{c^2} \\ \text{"c" : célérité des OEM (vide ou air)} \end{cases}$$

1° $\operatorname{div}(\vec{E}) = 0$

2° $\operatorname{div}(\vec{B}) = 0$

3° $\vec{\operatorname{rot}}(\vec{E}) = -\frac{\partial \vec{B}}{\partial t}$

4° $\vec{\operatorname{rot}}(\vec{B}) = \frac{1}{c^2} \cdot \frac{\partial \vec{E}}{\partial t}$

$$\operatorname{div}(\vec{E}) = \underbrace{\cancel{\frac{\partial E_x}{\partial t}}}_{E_x=0} + \underbrace{\cancel{\frac{\partial E_y}{\partial t}}}_{E_y=0} + \underbrace{\frac{\partial E_z}{\partial t}}_{0} = 0$$

car E_z ne dépend pas de (y,t)

Equation de Maxwell, 1° vérifiée

$$\operatorname{div}(\vec{B}) = \underbrace{\frac{\partial B_x}{\partial t}}_{B_x=0 \text{ car ne dépend pas de } x} + \underbrace{\cancel{\frac{\partial B_y}{\partial t}}}_{B_y=0} + \underbrace{\cancel{\frac{\partial B_z}{\partial t}}}_{B_z=0}$$

$(\vec{\nabla}\cdot\vec{B})$

$\operatorname{div}(\vec{B}) = 0$ ($\vec{B}$ transverse)

2° vérifiée.

$$\vec{\operatorname{rot}}(\vec{E}) = -\frac{\partial \vec{B}}{\partial t} : \begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} \frac{\partial E_z}{\partial y} - \cancel{\frac{\partial E_y}{\partial z}}^{0} \\ \cancel{\frac{\partial E_x}{\partial z}}^{0} - \cancel{\frac{\partial E_z}{\partial x}}^{0} \\ \cancel{\frac{\partial E_y}{\partial x}}^{0} - \cancel{\frac{\partial E_x}{\partial y}}^{0} \end{pmatrix}$$

$(\vec{\nabla} \wedge \vec{E})$

$$\operatorname{rot}(\vec{E}) = \begin{pmatrix} -kE_0 \sin(ky-\omega t) \\ 0 \\ 0 \end{pmatrix}$$

$$= -\frac{\partial \vec{B}}{\partial t} = -((-\omega)(-B_0 \sin(ky-\omega t))\vec{e_x}$$

$= -\omega B_0 \sin(ky-\omega t)\vec{e_x}$

$= -kc \frac{E_0}{c} \sin(ky-\omega t)\vec{e_x}$

$= -kE_0 \sin(ky-\omega t)\vec{e_x}$

$-\frac{\partial \vec{B}}{\partial t} = \vec{\operatorname{rot}}(\vec{E})$ 3° vérifiée.

Equation Générale : $\vec{E}(M,t) = 10^6 \cos\left(\frac{2\pi}{3} \cdot 10^7(\sqrt{3}x + y) - \omega t\right)\vec{e_z}$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\frac{4\pi}{3} \cdot 10^7} = \frac{3}{2} \cdot 10^{-7} \, m = 1{,}5 \times 10^{-7} \, m = 0{,}15 \times 10^{-6} \, m = 0{,}15 \, \mu m \text{ (UV)}$$

ω ?

On a $\omega = kc$

$$\omega = \frac{4\pi}{\cancel{3}} \cdot 10^7 \cdot \cancel{3} \cdot 10^8 = 4\pi \cdot 10^{15} \, rd.s^{-1} \simeq 12 \cdot 10^{15} \, rad.s^{-1} \quad (\pi \approx 3)$$

T ?

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi} \cdot 10^{-15} = 0{,}5 \times 10^{-15} \, s = 5 \times 10^{-16} \, s$$

$$f = \frac{\omega}{2\pi} = \frac{4\pi \cdot 10^{15}}{2\pi} = 2 \times 10^{15} \, Hz \text{ (onde lumineuse.}$$

3°) $\vec{\nabla} = i\vec{k}$ $\quad \frac{\partial}{\partial t} = -i\omega$

$E_0 = cB_0$

$B_0 = \frac{E_0}{c} = \frac{10^6}{3 \times 10^8} \simeq 0{,}33 \times 10^{-2}\,T$.

$\vec{rot}(\vec{E}) = \frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Leftrightarrow i\vec{k} \wedge \vec{E} = -(-i\omega \vec{B})$

$\vec{k} \wedge \vec{E} = \omega \vec{B} \Leftrightarrow \vec{B} = \frac{1}{\omega}(\vec{k} \wedge \vec{E}) = \frac{1}{\omega} \times \begin{pmatrix} k_x \\ k_y \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix}$